18. Strain at a point is such that $\varepsilon_{xx} = \varepsilon_{yy} = 0$, $\varepsilon_{zz} = -0.001$, $\varepsilon_{xy} = 0.006$, and $\varepsilon_{xz} = \varepsilon_{yz} = 0$. Note: You need not solve the eigen value problem for this question.

(a) Show that $n^1 = i + j$ and $n^2 = -i + j$ are principal directions of strain at this point.

(b) What is the third principal direction?

(c) Compute the three principal strains.

Solution:

(a) The strain matrix is

$$[\varepsilon] = \begin{bmatrix} 0 & 6 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times 10^{-3}$$

In order to show a direction $n$ is a principal direction, it is enough to show that $[\varepsilon] \cdot n = \lambda n$. After normalizing $n^1$ and $n^2$,

$$[\varepsilon] \cdot n^1 = \frac{10^{-3}}{\sqrt{2}} \begin{bmatrix} 0 & 6 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{10^{-3}}{\sqrt{2}} \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix} \parallel n^1$$

$$[\varepsilon] \cdot n^2 = \frac{10^{-3}}{\sqrt{2}} \begin{bmatrix} 0 & 6 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{10^{-3}}{\sqrt{2}} \begin{bmatrix} 6 \\ -6 \\ 0 \end{bmatrix} \parallel n^2$$

Thus, $n^1$ and $n^2$ are principal directions.

(b) From the orthogonal property of principal directions, the third principal direction can be found using the cross product as

$$n^3 = n^1 \times n^2 = \{0 \ 0 \ 1\}^T$$

Note that $n^3$ in the above equation is normalized.

(c) Since the third principal direction is parallel to the $z$-axis, $\varepsilon_{zz}$ is the third principal strain; i.e., $\varepsilon_3 = \varepsilon_{zz} = -0.001$. From Part (a), the principal strain $\varepsilon_1$ and $\varepsilon_2$ can be obtained because $[\varepsilon] \cdot n = \lambda n$. Thus, the three principal strains are

$$\varepsilon_1 = 0.006, \quad \varepsilon_2 = -0.001, \quad \varepsilon_3 = -0.006$$

Note that the three principal strains are reordered.