13. The uniaxial bar finite element equation can be used for other types of engineering problems, if proper analogy is applied. For example, consider the piping network shown in the figure. Each section of the network can be modeled using a finite element. If the flow is laminar and steady, we can write the equations for a single pipe element as:

\[
q_i = K(P_i - P_j) \\
q_j = K(P_j - P_i)
\]

where \( q_i \) and \( q_j \) are fluid flow at nodes \( i \) and \( j \), respectively; \( P_i \) and \( P_j \) are fluid pressure at nodes \( i \) and \( j \), respectively; and \( K \) is

\[
K = \frac{\pi D^4}{128 \mu L}
\]

where \( D \) is the diameter of the piper, \( \mu \) is the viscosity, and \( L \) is the length of the pipe. The fluid flow is considered positive away from the node. The viscosity of the fluid is \( 9 \times 10^{-4} \text{ Pa·s} \).

(a) Write the element matrix equation for the flow in the pipe element.

(b) The net flow rates into nodes 1 and 2 are 10 and 15 m\(^3\)/s, respectively. The pressures at the nodes 6, 7, and 8 are all zero. The net flow rate into the nodes 3, 4, and 5 are all zero. What is the outflow rate for elements 4, 6, and 7?

<table>
<thead>
<tr>
<th>Elem</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>D(mm)</td>
<td>40</td>
<td>40</td>
<td>50</td>
<td>25</td>
<td>40</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>L(m)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

**Solution:**

(a) The element matrix equation becomes:

\[
K \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} P_i \\ P_j \end{bmatrix} = \begin{bmatrix} q_i \\ q_j \end{bmatrix}
\]

Comparing to uniaxial bar elements, the nodal pressure corresponds to the nodal displacement, while the flow rate to the nodal force.

(b) Using the above table, \( K \) for each element can be calculated as
Then, the seven elements are assembled to yield the following global matrix:

\[
10^{-5}
\begin{bmatrix}
6.98 & 0 & -6.98 & 0 & 0 & 0 & 0 \\
0 & 6.98 & -6.98 & 0 & 0 & 0 & 0 \\
-6.98 & -6.98 & 31 & -17.04 & 0 & 0 & 0 \\
0 & 0 & -17.04 & 20.796 & -3.49 & -0.266 & 0 \\
0 & 0 & 0 & -3.49 & 4.2 & 0 & -0.355 \\
0 & 0 & 0 & 0 & -0.266 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.355
\end{bmatrix}
\]

Since the pressure at Nodes 6, 7, and 8 are zero, we can delete the rows and columns corresponding to these nodes. Then, the global matrix equation becomes

\[
10^{-5}
\begin{bmatrix}
6.98 & 0 & -6.98 & 0 & 0 \\
0 & 6.98 & -6.98 & 0 & 0 \\
-6.98 & -6.98 & 31 & -17.04 & 0 \\
0 & 0 & -17.04 & 20.796 & -3.49 \\
0 & 0 & 0 & -3.49 & 4.2 \\
0 & 0 & 0 & 0 & -0.266
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5 \\
P_6
\end{bmatrix}
= 
\begin{bmatrix}
10 \\
15 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

The above equation can be solved for unknown nodal pressures as

\[
\{P_1, P_2, P_3, P_4, P_5\} = \{3.21, 3.28, 3.07, 2.92, 2.43\} \text{ MPa}
\]

Thus, the outflow rate can be calculated as

\[
Q^{(4)} = K^{(4)}(P_4 - P_6) = 0.266 \times 10^{-5}(2.92 \times 10^6) = 7.767 \text{ m}^3/\text{s}
\]

\[
Q^{(6)} = K^{(6)}(P_5 - P_7) = 0.355 \times 10^{-5}(2.43 \times 10^6) = 8.627 \text{ m}^3/\text{s}
\]

\[
Q^{(7)} = K^{(7)}(P_5 - P_8) = 0.355 \times 10^{-5}(2.43 \times 10^6) = 8.627 \text{ m}^3/\text{s}
\]