13. Consider the following displacement field in a plane solid:

\[ u(x, y) = 0.04 - 0.01x + 0.006y \]
\[ v(x, y) = 0.06 + 0.009x + 0.012y \]

(a) Compute the strain components \( \varepsilon_{xx} \), \( \varepsilon_{yy} \), and \( \gamma_{xy} \). Is this a state of uniform strain?

(b) Determine the principal strains and their corresponding directions. Express the principal strain directions in terms of angles the directions make with the \( x \)-axis.

(c) What is the normal strain at Point \( O \) in a direction 45° to the \( x \)-axis?

Solution:

(a) Strain components:

\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} = -0.01 \]
\[ \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0.012 \]
\[ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.009 + 0.006 = 0.015 \]

Yes, this is a state of uniform strain, because the strains are independent of position \( x, y, z \).

(b) Principal strains and principal directions.

\[ \varepsilon_{xy} = \frac{1}{2} \gamma_{xy} = 0.0075 \]

\[ \varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix} \]

Find the eigen values (principal strains) and eigen vectors (principal direction) by solving the eigen value problem:

\[ \begin{bmatrix} -0.01 - \lambda & 0.0075 \\ 0.0075 & 0.012 - \lambda \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

The above equation yields two principal strains: \( \varepsilon_1 = \lambda_1 = -0.01231 \) and \( \varepsilon_2 = \lambda_2 = 0.01431 \). The principal direction corresponding to the first principal strain is

\[ \mathbf{n}^{(1)} = \begin{bmatrix} -0.9556 \\ 0.2948 \end{bmatrix}, \]

The angle the direction makes with the \( x \)-axis can be found from the relation

\[ \cos \theta = -0.9556, \sin \theta = 0.2948. \]

Solving \( \theta \approx 163^\circ \)

The principal direction corresponding to the second principal strain is
\[ \mathbf{n}^{(2)} = \begin{bmatrix} 0.2948 & 0.9556 \end{bmatrix}, \]

and the angle is found to be \( \theta \approx 73^\circ \)

(c)

Strain at point O

\[ \mathbf{\varepsilon} = \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix}, \]

direction vector

\[ \mathbf{n} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \]

Thus the normal strain in the direction of \( \mathbf{n} \) becomes

\[ \varepsilon_{45^\circ} = \mathbf{n} \cdot \mathbf{\varepsilon} \cdot \mathbf{n} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} -0.01 & 0.0075 \\ 0.0075 & 0.012 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 0.0085 \]