21. Consider the tapered bar in Problem 16. Use the Rayleigh-Ritz method to solve the same problem. Assume the displacement in the form of

\[ u(x) = (x - 1)(c_1x + c_2x^2). \]

**Solution:**

The assumed displacements satisfy the essential BCs \( u(0) = u(1) = 0 \). The assumed displacements take the form:

\[ u(x) = (x - 1)(c_1x + c_2x^2) \]

Then, the strain is given by

\[ \varepsilon_{xx} = \frac{du}{dx} = c_1(2x - 1) + c_2(3x^2 - 2x) \]

The strain energy in the bar becomes

\[ U = \frac{E}{2} \int_0^1 A(x) \left[ c_1(2x - 1) + c_2(3x^2 - 2x) \right]^2 dx \]

where \( A(x) \) is the area of cross-section defined as

\[ A(x) = \pi 0.05^2(1 - 0.8x)^2 \]

The potential energy of the distributed load \( f = 10,000 \) N/m is

\[ V = -\int_0^1 f u(x) dx = -10,000 \int_0^1 (x - 1)(c_1x + c_2x^2) dx \]

The total potential energy \( \Pi(c_1, c_2) = U(c_1, c_2) + V(c_1, c_2) \). The principle of minimum potential energy requires

\[ \frac{\partial \Pi}{\partial c_1} = \frac{\partial \Pi}{\partial c_2} = 0 \]

The results in a set of equations in \( c_1 \) and \( c_2 \), as

\[ \begin{bmatrix} .1194 & .0346 \\ .0346 & .0221 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 166.667 \\ 83.333 \end{bmatrix} \]

The above equation can be solved for unknown coefficients, as

\[ c_1 = 0.000555, \quad c_2 = 0.0029 \]

Thus, the approximate displacement becomes
u(x) = (x - 1)(.000555x + .0029x^2)

The axial member force becomes

\[ P(x) = AE \frac{du}{dx} = A(x)E \left[ c_1(2x - 1) + c_2(3x^2 - 2x) \right] \]